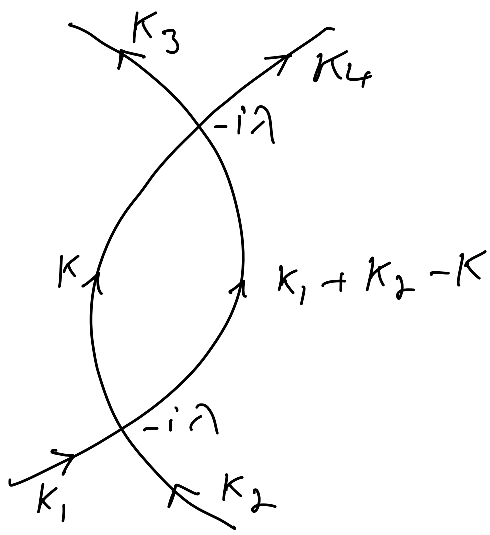
§4. Renormalization

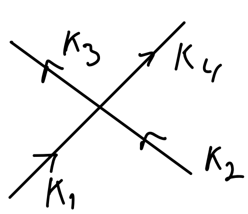
§4.1 Infinities and cutoffs

Feynman graph calculations often lead to divergent integrals

→ Recall the diagram for scattering two scalar fields in φ^4-theory:



→ gives an order λ^2 correction to



The corresponding amplitude is given by

$$\mathcal{M} = \frac{1}{2}(-i\lambda)^2 i^2 \int \frac{d^4k}{(2\pi)^4} \frac{1}{k^2-m^2+i\varepsilon} \frac{1}{(k_1+k_2-k)^2-m^2+i\varepsilon} \qquad (1)$$

$\rightarrow$ the integrand goes as $\frac{1}{k^4}$ for $k \gg 0$ leading to a logarithmic divergence:

$$\int \frac{d^4k}{(2\pi)^4} \frac{1}{k^4} \sim \lim_{\Lambda \to \infty} \log\left(\frac{\Lambda^2}{k^2}\right), \quad k = k_1 + k_2 \quad (*)$$

"ultraviolet divergence"
(divergence associated to large frequency modes)

modern view: QFT is an effective low energy theory, valid up to some energy scale Λ
($\Lambda \sim \frac{1}{a}$, a: lattice spacing on "mattres" of spacetime)

Λ is known as "cutoff"

Let's verify equation $(*)$:

we begin with the convergent integral

$$I = \int \frac{d^4k}{(2\pi)^4} \frac{1}{(k^2 - m^2 + i\varepsilon)^3}$$

$$= \int \frac{d^3k}{(2\pi)^3} \int \frac{dk_0}{2\pi} \frac{1}{[k_0^2 - (\vec{k}^2 + m^2) + i\varepsilon]^3}$$

$\longrightarrow$ perform a Wick rotation:

$$\int_{-\infty}^{\infty} dk_0\, f(k_0) = \int_{-i\infty}^{i\infty} dk_0\, f(k_0) = i\int_{-\infty}^{\infty} dk_4\, f(ik_4)$$

where we defined $k_0 = ik_4$

$$\longrightarrow\quad I = i(-1)^3 \int \frac{d_E^3 k}{(2\pi)^4} \frac{1}{(k_E^2 + m^2)^3}$$

where $d_E^4 k$ is now the integration element in Euclidean 4-dim space and $k_E^2 := k_4^2 + \vec{k}^2$

Now note that

$$\int \frac{d^D k_E}{(2\pi)^D} f(k_E) = \frac{S_D}{(2\pi)^D} \int_0^\infty dk\, k^3 f(k)$$

where S_D is the volume of the $(D-1)$-dim unit sphere: $S_D = \frac{2\pi^{D/2}}{\Gamma(D/2)}$ (exercise)

giving in our case

$$I = i(-1)^3 \frac{2\pi^2}{\Gamma(2)} (2\pi)^{-4} \int_0^\infty dk\, k^3 f(k)$$

$$= \frac{-i}{16\pi^2} \int_0^\infty dk\, k^3 \frac{1}{(k^2+m^2)^3}$$

$$= \frac{-i}{16\pi^2} \lim_{\Lambda\to\infty} \left[-\frac{k^2+m^2}{2(k^2+m^2)^2} \right]_0^\Lambda = \frac{-i}{32\pi^2} \frac{1}{m^2}$$

Now consider

$$\hat{I} = \int \frac{d^4k}{(2\pi)^4} \frac{1}{(k^2-m^2+i\varepsilon)^2} \qquad (2a)$$

$\longrightarrow$ can show similarly

$$I(\Lambda^2, m^2) = \frac{i}{16\pi^2} \log\left(\frac{\Lambda^2}{m^2}\right) \qquad (2b)$$

as a check, differentiate both sides with respect to m^2:

$$\text{l.s.} = 2\int \frac{d^4k}{(2\pi)^4} \frac{1}{(k^2-m^2+i\varepsilon)^3}$$

$$= \frac{-i}{16\pi^2} \frac{1}{m^2}$$

$$\text{r.s.} = \frac{d}{dm^2}\left(\frac{i}{16\pi^2}\left[\log\left(\frac{\Lambda^2}{m^2}\right) - 1 + \cdots\right]\right)$$

$$= \frac{+i}{16\pi^2} \frac{m^2}{\Lambda^2} \frac{d}{dm^2}\left(\frac{\Lambda^2}{m^2}\right)$$

$$= \frac{-i}{16\pi^2} \frac{1}{m^2}$$

Now let's consider integral (1): using the identity

$$\frac{1}{xy} = \int_0^1 d\alpha \frac{1}{[\alpha x + (1-\alpha)y]^2},$$

it can be rewritten as

$$\mathcal{M} = \frac{1}{2}(-i\lambda)^2 i^2 \int \frac{d^4k}{(2\pi)^4} \int_0^1 d\alpha \, \frac{1}{D}$$

where $(K := k_1 + k_2)$

$$D = \alpha(K-k)^2 + (1-\alpha)k^2 - m + i\varepsilon$$
$$= (k-\alpha K)^2 + \alpha(1-\alpha)K^2 - m^2 + i\varepsilon$$

shift the integration variable $k \mapsto k + \alpha K$

giving

$$\mathcal{M} = \frac{1}{2}(-i\lambda)^2 i^2 \int \frac{d^4k}{(2\pi)^4} \int_0^1 d\alpha \, \frac{1}{[k^2 - c^2 + i\varepsilon]^2}$$

where $c^2 = m^2 - \alpha(1-\alpha)K^2$

Using the results (2a), (2b), we finally conclude

$$\mathcal{M} = \frac{i\lambda^2}{32\pi^2} \int_0^1 d\alpha \log\left(\frac{\Lambda^2}{m^2 - \alpha(1-\alpha)K^2 - i\varepsilon}\right) \quad (3)$$

→ result is indeed logarithmically divergent!

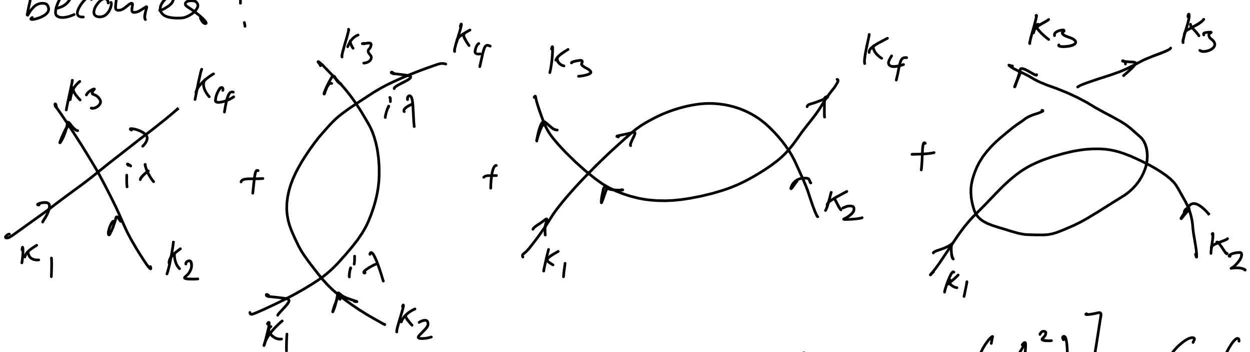
performing the integral in (3) and assuming $m^2 \ll K^2$, we get

$$\mathcal{M} \sim \underset{\text{const.}}{C} \log\left(\frac{\Lambda^2}{K^2}\right) \qquad \text{(exercise)}$$

It is convenient to define kinematic variables $s := K^2 = (k_1+k_2)^2$, $t := (k_1-k_3)^2$,

$u := (k_1-k_4)^2$,

the full scattering amplitude to order $\mathcal{O}(\lambda^3)$ becomes:



$$= -i\lambda + iC\lambda^2\left[\log\left(\frac{\Lambda^2}{s}\right) + \log\left(\frac{\Lambda^2}{t}\right) + \log\left(\frac{\Lambda^2}{u}\right)\right] + \mathcal{O}(\lambda^3) \tag{4}$$

that is $\mathcal{M}$ depends logarithmically on the cutoff

Renormalization:

In order for the amplitude $\mathcal{M}$ to be sensible (for an experiment), it should not depend on the cutoff Λ, this easy to fix

$\rightarrow$ change Λ as $e^{\varepsilon}\Lambda$

$\rightarrow$ $\log\Lambda \mapsto \log\Lambda + \varepsilon$

and thus

$$\delta \mathcal{M} = -i\delta\lambda + iC\lambda^2 3(2\varepsilon) + O(\lambda^3)$$

and so

$$\delta \mathcal{M} = 0$$

implies

$$\delta\lambda = 6C\lambda^2\varepsilon + O(\lambda^3)$$
$$= 6C\lambda^2 \delta \log \Lambda + O(\lambda^3)$$

finally,

$$\Lambda \frac{d\lambda}{d\Lambda} = \frac{d\lambda}{d(\log\Lambda)} = 6C\lambda^2 + O(\lambda^3)$$

We will continue next time to see how the now Λ-indep. amplitud $\mathcal{M}$ is compared to a measurement (experiment)